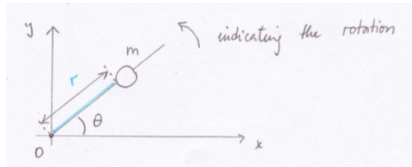


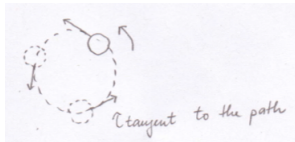
Symmetries 2 - Rotations in Space

This symmetry is about the isotropy of space, i.e. space is the same in all orientations. Thus, if we continuously rotated an entire system in space, we expect the system to remain unchanged.

Rotations mean things going round and round. If we allow the particle of mass m to rotate about an axis, the interesting variables are the distance r from the axis of rotation and the angle θ from some horizontal axis to describe by how much we have rotated the position of the particle.



The first time derivative of angle θ gives the angular velocity $\omega = \frac{d\theta}{dt}$. The speed in the direction of movement is then given by the distance r from the axis times the angular velocity $v_t = r \frac{d\theta}{dt}$, keeping r fixed. The notation v_t is chosen to refer to the component tangent to the path of the velocity of the particle.



Symmetries 2 - Rotations in Space

In analogy with linear momentum that describes how fast the particle moves away from the origin, we can define angular momentum, denoted often as $\underline{L}$, to describe how much the particle is going around the origin.

A particle of mass m that goes round a circle of radius $r = a$ with angular velocity $\frac{d\theta}{dt}$ has angular momentum $L = ma^2 \frac{d\theta}{dt}$.

In direct analogy with the translation invariance of linear momentum, we see that the **angular momentum** remains **unchanged** under a **continuous rotation** of the frame.

Let $\theta' = \theta + \alpha$, where α is some constant angle, then in the primed system $L_{\theta'} = mr^2 \frac{d\theta'}{dt} = mr^2 \frac{d(\theta + \alpha)}{dt} = mr^2 \frac{d\theta}{dt} = L_{\theta}$, since $\frac{d\alpha}{dt} = 0$. So, **angular momentum is rotation invariant**.

The rate of change of angular momentum equals the external torque τ :

$$\frac{dL}{dt} = \tau$$

Where the torque describes how much a force rotates an object about an axis.

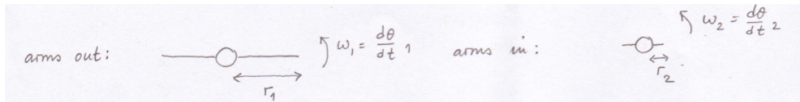
Thus, again for an isolated system i.e. a system without an external torque, the angular momentum is conserved

$$\frac{dL}{dt} = 0$$

Therefore, the mechanical properties of an isolated system i.e. a system with no external fields to break the symmetry, do not change when the entire system is rotated in any manner in space.

Symmetries 2 - Rotations in Space

For systems in an external potential $u = u(r, \theta, t)$ the angular momentum is conserved if the potential has **rotational symmetry**, i.e. the potential does not depend on the angle θ and thus $\frac{\partial u}{\partial \theta} = 0$. Example: an ice skater spinning in a gravitational field. The angular momentum $L = mr^2 \frac{d\theta}{dt}$ is conserved because the gravitational potential is not dependent on θ . Schematically seen from above:



Angular momentum conserved $\rightarrow L_1 = L_2$, but since $m_1 = m_2 = m$ and $r_1 > r_2$, we get $\omega_2 > \omega_1$, i.e. the skater spins faster with arms in.

Symmetries 3 - Time translations

Homogeneity of time, i.e. time is the same at every time, is the symmetry responsible for energy conservation.

Let $t \rightarrow t + \alpha$, with α some constant, to represent a continuous time translation. Consider a one dimensional system of a particle of mass m with kinetic energy $\frac{m}{2} \left(\frac{dx}{dt} \right)^2$ and potential energy $U(x)$. If $U = U(t)$ explicitly, then E is not conserved.

The total energy is $E = \frac{m}{2} \left(\frac{dx}{dt} \right)^2 + U(x)$.

Differentiating the above equation with respect to time using the product rule, we obtain for the kinetic energy

$$\frac{d}{dt} \left(\frac{m}{2} \left(\frac{dx}{dt} \right)^2 \right) = m \left(\frac{dx}{dt} \right) \left(\frac{d^2x}{dt^2} \right)$$

For the potential energy, remember x depends on t , so $\frac{dU}{dt} = \frac{dU}{dx} \frac{dx}{dt}$ by the chain rule. Then;

$$\frac{dE}{dt} = \frac{dx}{dt} \left[m \frac{d^2x}{dt^2} + \frac{dU}{dx} \right]$$

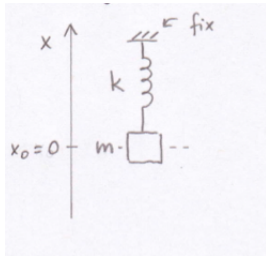
Recognising Newton's Second Law of motion, $m \frac{d^2x}{dt^2} = \frac{dp}{dt} = -\frac{dU}{dx}$, you see that

$$\frac{dE}{dt} = 0$$

So energy is conserved! Thus, isolated systems have no explicit time dependence $\rightarrow$ energy is conserved. Conservative systems, i.e. systems with constant external field $\rightarrow$ energy is conserved. For example, for the particle of mass m projected in the presence of an external field, gravity, the total energy $E = T + U =$ kinetic energy + potential energy is conserved.

Example: Simple Harmonic Oscillator

Suppose we have a spring of **spring constant k** fixed in space at one end and has a **mass m** at the other end. The mass will stretch the spring until it reaches a vertical position of equilibrium, i.e. a position $x_0 = 0$ at which the restoring spring force, given by Hooke's law $F = -kx$, exactly balances the gravitational force $F = mg$.



k, m are constants. Newton's Second Law of motion for small oscillations about the position of equilibrium x_0 gives

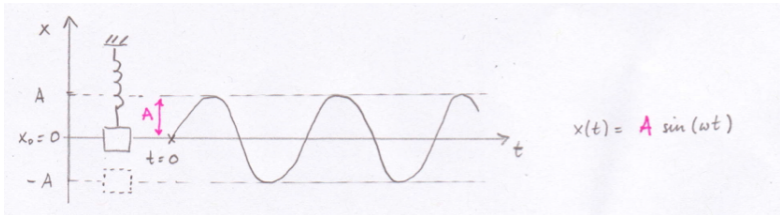
$$m \frac{d^2 x}{dt^2} = -kx$$

Where x is small and it measures the vertical distances from $x_0 = 0$.

Example: Simple Harmonic Oscillator

If at time $t = 0$ the mass is at the position $x_0 = 0$, then the solution to this differential equation is given by $x(t) = A \sin(\omega t)$, where A is the amplitude of the oscillations and ω is the angular frequency of these oscillations such that $\omega^2 = \frac{k}{m}$.

We can visualise this motion by imagining a pen being attached to the mass that leaves a mark on the surface behind. If we then stretch that out over time, we would see something like this:

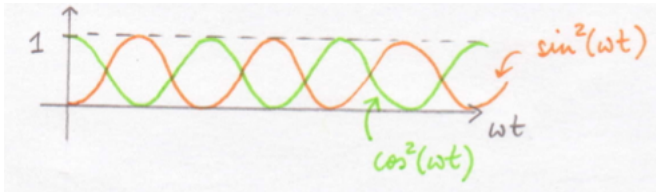


The total energy of this system is given by $E = \frac{m}{2} \left(\frac{dx}{dt} \right)^2 + \frac{kx^2}{2}$. Here we have **position** $x(t) = A \sin(\omega t)$ and we can find speed $\frac{dx}{dt} = A\omega \cos(\omega t)$. Thus we have for the energy $E = \frac{m}{2} A^2 \omega^2 \cos^2(\omega t) + \frac{k}{2} A^2 \sin^2(\omega t)$. Recalling that $\omega^2 = \frac{k}{m}$ and factoring out the common factor, we have:

$$E = A^2 \frac{k}{2} [\cos^2(\omega t) + \sin^2(\omega t)]$$

Example: Simple Harmonic Oscillator

At this point it still looks like we have a time dependence, but looks can be deceptive. Let us graph $\cos^2(\omega t)$ and $\sin^2(\omega t)$:



Adding these two graphs, we see that the sum for any t is always 1. In fact, this result $\sin^2(\omega t) + \cos^2(\omega t) = 1$ is true for any angle, i.e. if we let $t \rightarrow t' = t + \alpha$, we still have $\sin^2(\omega t') + \cos^2(\omega t') = 1$. Hence, $E = A^2 \frac{k}{2} = \text{constant}$ and

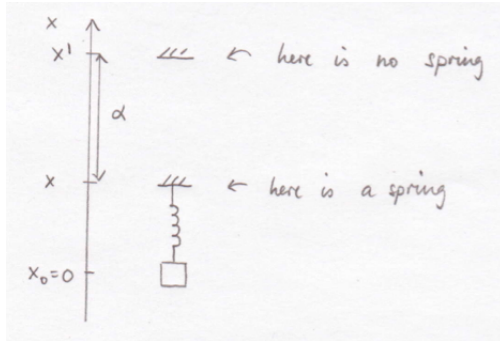
$$\frac{dE}{dt} = 0$$

and so energy is conserved for an harmonic oscillator.

Example: Simple Harmonic Oscillator

What about momentum conservation?

The spring is fixed at an arbitrary, but once chosen fixed point in space, and consequently we were looking at motion about a fixed point x_0 . If we make a linear translation in space letting $x \rightarrow x' = x + \alpha$, we notice that there is no spring!



So momentum is not conserved.

Summary

Whenever there is a continuous symmetry, there is a conserved quantity.

- Emmy Noether 1918.

- Continuous parallel translations in space lead to momentum conservation.
- Continuous rotations in space lead to angular momentum conservation.
- Continuous translations in time lead to energy conservation.